Stochastic Present Value

&

Jensen’s Inequality

by

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Present Value

Present value of earnings for year $t$ is the amount that would have to be invested today with an annual rate of return $d$, also called the discount rate, so as to compound into becoming $Earnings_t$ after $t$ years:

$$Present\ Value\ of\ Earnings_t \cdot (1 + d)^t = Earnings_t$$

\[\iff\]

$$Present\ Value\ of\ Earnings_t = \frac{Earnings_t}{(1 + d)^t}$$

The present value for $n$ years is the sum:

$$Present\ Value = \sum_{t=1}^{n} \frac{Earnings_t}{(1 + d)^t}$$
Jensen’s Inequality

Let $X$ be a stochastic variable and let $\varphi$ be a convex function, then Jensen’s inequality states that:

$$E[\varphi(X)] \geq \varphi(E[X])$$

This becomes a strict inequality if $\varphi$ is strictly convex and $Var[X] > 0$.

Present value calculations are exponential $X^t$ with stochastic variables $X \geq 0$ and exponents $t = 1, 2, 3, ...$ This is a convex function, so:

$$E[X^t] \geq E[X]^t$$
Jensen’s Inequality, Example

$X$ is either 0.9 or 1.5. Consider $X^4$

$$E[X^4] = \frac{0.9^4 + 1.5^4}{2}$$

$$= \frac{0.6561 + 5.0625}{2} \approx 2.86$$

$$E[X]^4 = \left(\frac{0.9 + 1.5}{2}\right)^4 \approx 2.07$$

So Jensen’s inequality is satisfied:

$$E[X^4] \geq E[X]^4$$
**Stochastic Discount Rate**

Let $D$ be a stochastic discount rate. The stochastic present value is:

$$PV = \sum_{t=1}^{n} \frac{Earnings_t}{(1 + D)^t}$$

This uses an exponential function, so according to Jensen’s inequality:

$$E[PV] = E \left[ \sum_{t=1}^{n} \frac{Earnings_t}{(1 + D)^t} \right] \geq \sum_{t=1}^{n} \frac{Earnings_t}{(1 + E[D])^t}$$

So using the mean discount rate may underestimate the present value.
Stochastic Growth Rate

Let $G$ be a stochastic growth rate. The stochastic present value is:

$$PV = \sum_{t=1}^{n} Earnings \cdot \left( \frac{1 + G}{1 + d} \right)^t$$

This uses an exponential function, so according to Jensen’s inequality:

$$E[PV] = E \left[ \sum_{t=1}^{n} Earnings \cdot \left( \frac{1 + G}{1 + d} \right)^t \right] \geq Earnings \cdot \sum_{t=1}^{n} \left( \frac{1 + E[G]}{1 + d} \right)^t$$

So using the mean growth rate may underestimate the present value.
Stochastic Growth & Discount Rate

Let $G$ and $D$ be stochastic growth and discount rates, respectively.

The stochastic present value is:

$$PV = \sum_{t=1}^{n} Earnings \cdot \left( \frac{1 + G}{1 + D} \right)^t$$

This uses an exponential function, so according to Jensen’s inequality:

$$E[PV] = E \left[ \sum_{t=1}^{n} Earnings \cdot \left( \frac{1 + G}{1 + D} \right)^t \right] \geq Earnings \cdot \sum_{t=1}^{n} \left( \frac{1 + E[G]}{1 + E[D]} \right)^t$$
Stochastic Discount Rate, Example

Let $D$ be a stochastic discount rate that can be either 2%, 10% or 14%.

The average discount rate is $E[D] \approx 8.7\%$.

Let $Earnings_t = 1$ for convenience.

The mean present value for the first ten years is:

$$E \left[ \sum_{t=1}^{10} \frac{1}{(1 + D)^t} \right] \approx 6.78 \geq 6.51 \approx \sum_{t=1}^{10} \frac{1}{(1 + E[D])^t}$$

The actual mean (left) is 4% greater than the estimate (right).
Stochastic Growth Rate, Example

Let $G$ be a stochastic growth rate that can be either -10%, 5% or 20%.

The average growth rate is $E[G] = 5\%$.

Let discount rate be $d = 10\%$.

The mean present value for the first ten years is:

$$E \left[ \sum_{t=1}^{10} \left( \frac{1 + G}{1 + d} \right)^t \right] \approx 9.45 \geq 7.81 \approx \sum_{t=1}^{10} \left( \frac{1 + E[G]}{1 + d} \right)^t$$

The actual mean (left) is 21% greater than the estimate (right).
**Stochastic Growth & Discount Rate, Example**

Let $G$ be a stochastic growth rate that can be either -10%, 5% or 20%.

Let $D$ be a stochastic discount rate that can be either 2%, 10% or 14%.

Assume they are independent so there are 9 combinations of $G$ and $D$.

The mean present value for the first ten years is:

$$E \left[ \sum_{t=1}^{10} \left( \frac{1 + G}{1 + D} \right)^t \right] \approx 10.67 \geq 8.32 \approx \sum_{t=1}^{10} \left( \frac{1 + E[G]}{1 + E[D]} \right)^t$$

The actual mean (left) is 28% greater than the estimate (right).
Greater Variance, Example

Let $G$ be a stochastic growth rate that can be either -15%, 5% or 25%. Let $D$ be a stochastic discount rate that can be either 2%, 10% or 14%. Assume they are independent so there are 9 combinations of $G$ and $D$.

The mean present value for the first ten years is:

$$E \left[ \sum_{t=1}^{10} \left( \frac{1 + G}{1 + D} \right)^t \right] \approx 12.34 \geq 8.32 \approx \sum_{t=1}^{10} \left( \frac{1 + E[G]}{1 + E[D]} \right)^t$$

The actual mean (left) is 48% greater than the estimate (right).
Conclusion

• The mean present value may be underestimated when it is calculated from the mean growth and discount rates.
• The magnitude of the error depends on the variance of the stochastic growth and discount rates.
Further Reading

This lecture is based on the paper:

- Monte Carlo Simulation in Financial Valuation

Authored by Magnus Erik Hvass Pedersen.

Available on the internet:

www.Hvass-Labs.Org